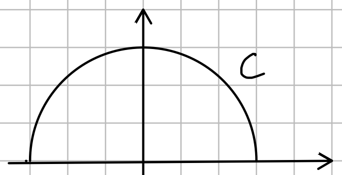
1. (Stewart 16.2/Example 3) A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.



density : $\rho(x,y) = K(1-y)$

mass : $m(C)$ $\displaystyle\int_C \rho(x,y)\, d(x,y) = \int_0^\pi \rho(\cos(t), \sin(t)) \cdot \underbrace{\|r'(t)\|}_{1}\, dt = \int_0^\pi K(1-\sin(t))\, dt = K\Big[t + \cos(t)\Big]_{t=0}^{\pi} = K(\pi - 2)$

parametrization of C : $r(t) = (\cos(t), \sin(t))$ $\quad t \in [0,\pi]$

$r'(t) = (-\sin(t), \cos(t))$ $\quad \|r'(t)\| = 1$

center of mass: $\displaystyle\bar{x} = \frac{1}{m(C)} \cdot \int_C x \cdot \rho(x,y)\, d(x,y) = \frac{1}{m(C)} \underbrace{\int_C x(1-y)\, d(x,y)}_{= 0 \;\leftarrow\; \text{by symmetry}} = 0$

$$\bar{y} = \frac{1}{m(C)} \cdot \int_C y \cdot \overbrace{\rho(x,y)}^{K\cdot(1-y)}\, d(x,y) = \frac{K}{m(C)} \cdot \int_0^\pi \sin(t) - \overbrace{\sin^2(t)}^{\frac{1-\cos(2t)}{2}}\, dt = \frac{K}{m(C)} \cdot \Big[-\cos(t) - \frac{t}{2} + \frac{\sin(2t)}{4}\Big]_{t=0}^{\pi} = \frac{K}{m(C)}\Big(2 - \frac{\pi}{2}\Big) =$$

$$= \frac{4-\pi}{2(\pi-2)}$$

center of mass : $(\bar{x}, \bar{y}) = \left(0, \dfrac{4-\pi}{2(\pi-2)}\right)$

2. (Stewart 16.2/Example 8, Exercise 23) Evaluate the integral $\int_C \underline{F} \cdot d\underline{r}$ where C is given by the vector function $\underline{r}$.

a) $\underline{F}(x,y,z) := xy\underline{i} + yz\underline{j} + zx\underline{k}$, $\quad \underline{r}(t) := (t, t^2, t^3) \quad (0 \leq t \leq 1)$

b) $\underline{F}(x,y,z) := \sin(x)\underline{i} + \cos(y)\underline{j} + xz\underline{k}$, $\quad \underline{r}(t) := t^3\underline{i} - t^2\underline{j} + t\underline{k} \quad (0 \leq t \leq 1)$

a) $F(x,y,z) = (xy, yz, xz)$ $\quad ((x,y,z) \in \mathbb{R}^3)$ $\quad C: r(t) = (t, t^2, t^3)$ $\quad (t \in [0,1])$

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt = \int_0^1 (t^3, t^5, t^4) \cdot (1, 2t, 3t^2)\, dt = \int_0^1 t^3 + \underbrace{2t^6 + 3t^6}_{5t^6}\, dt = \Big[\frac{t^4}{4} + \frac{5t^7}{7}\Big]_{t=0}^{1} =$$

$r'(t) = (1, 2t, 3t^2)$, $\quad F(r(t)) = (t^3, t^5, t^4)$

$$= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

b) $F(x,y,z) := (\sin(x), \cos(y), xz)$ $\quad ((x,y,z) \in \mathbb{R}^3)$ $\quad C: r(t) = (t^3, -t^2, t)$ $\quad (t \in [0,1])$

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt = \int_0^1 3t^2 \sin(t^3) - 2t\cos(t^2) + t^4\, dt = \Big[-\cos(t^3) - \sin(t^2) + \frac{t^5}{5}\Big]_{t=0}^{1} =$$

$F(r(t)) = (\sin(t^3), \cos(t^2), t^4)$ $\quad r'(t) = (3t^2, -2t, 1)$

$$= \frac{6}{5} - \cos(1) - \sin(1)$$

3. (Stewart 16.2/45) The position of an object with mass m at time t is $\underline{r}(t) := at^2\underline{i} + bt^3\underline{j}$, $\quad (0 \leq t \leq 1)$.

a) What is the force acting on the object at time t?

b) What is the work done by the force during the time interval $0 \leq t \leq 1$?

a) $C: \quad r(t) = (at^2, bt^3)$ $\quad (t \in [0,1])$ $\quad r'(t) = (2at, 3bt^2)$

$F(r(t)) = m \cdot r''(t) = m(2a, 6bt)$

b) Work $\displaystyle\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt = m \cdot \int_0^1 4a^2 t + 18b^2 t^3\, dt = m \cdot \Big[2a^2 t^2 + \frac{9b^2}{2} t^4\Big]_{t=0}^{1} = \Big(2a^2 + \frac{9b^2}{2}\Big) \cdot m$

4. (Stewart 16.3/ Example 5) Let

$$\underline{F}(x,y,z) := y^2\underline{i} + (2xy + e^{3z})\underline{j} + 3ye^{3z}\underline{k} \qquad ((x,y,z) \in \mathbb{R}^3)$$

Find a function f such that $\nabla f = \underline{F}$.

$\partial_1 f(x,y,z) = y^2 \implies f(x,y,z) = xy^2 + g(y,z)$

$\partial_2 f(x,y,z) = 2xy + e^{3z} \implies f(x,y,z) = xy^2 + y \cdot e^{3z} + h(z)$

$\partial_3 f(x,y,z) = 3y \cdot e^{3z} \implies f(x,y,z) = xy^2 + y \cdot e^{3z} + C \qquad (C \in \mathbb{R})$

5. (Stewart 16.3/21) Let C be the line segment from $(2,-3,1)$ to $(-5,1,2)$ and

$$\underline{F}(x,y,z) := 2xy\underline{i} + (x^2 + 2yz)\underline{j} + y^2\underline{k} \qquad ((x,y,z) \in \mathbb{R}^3).$$

Find a function f such that $\nabla f = \underline{F}$ and use this to evaluate $\int_C \underline{F} \cdot d\underline{r}$ along the curve C.

$\underline{r}$ is the parametrization of C $\qquad \underline{r}(t) = (1-t)(2,-3,1) + t(-5,1,2) \qquad (t \in [0,1])$

$\underline{r}(0) = (2,-3,1) \qquad \underline{r}(1) = (-5,1,2)$

$\underline{F}(x,y,z) = (2xy,\ x^2+2yz,\ y^2)$

$\partial_1 f(x,y,z) = 2xy \implies f(x,y,z) = x^2y + g(y,z)$

$\partial_2 f(x,y,z) = x^2 + 2yz \implies f(x,y,z) = x^2y + y^2z + h(z)$

$\partial_3 f(x,y,z) = y^2 \implies f(x,y,z) = x^2y + y^2z + C \qquad (\text{let } C := 0)$

$$\int_C \underline{F} \cdot d\underline{r} = \int_C \nabla f \cdot d\underline{r} = f(\underline{r}(1)) - f(\underline{r}(0)) = f(-5,1,2) - f(2,-3,1) = 25 + 2 - (-12 + 9) = \underline{\underline{30}}$$

6. (Stewart 16.3/25) Let C be any path from $(1,0)$ to $(2,1)$. Show that the following line integral is independent of path and evaluate the integral.

$$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})dy$$

$\underline{F}(x,y) = (2x \cdot e^{-y},\ 2y - x^2 \cdot e^{-y}) \qquad ((x,y) \in \mathbb{R}^2)$

$\partial_2 F_1(x,y) = -2x \cdot e^{-y}$

$\qquad \|$

$\partial_1 F_2(x,y) = -2x \cdot e^{-y}$

$\implies \underline{F}$ is conservative $\iff \exists f: \mathbb{R}^2 \to \mathbb{R}$ s.t. $\nabla f = \underline{F}$

$\partial_1 f(x,y) = 2x \cdot e^{-y} \implies f(x,y) = x^2 \cdot e^{-y} + g(y)$

$\partial_2 f(x,y) = 2y - x^2 \cdot e^{-y} \implies f(x,y) = x^2 \cdot e^{-y} + y^2 (+C)$

$$\int_C \underline{F} \cdot d\underline{r} = f(2,1) - f(1,0) = \frac{4}{e} + 1 - (1+0) = \frac{4}{e}$$

7. (Stewart 16.3/35) Show that if the vector field $F = P\underline{i} + Q\underline{j} + R\underline{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

$\underline{F} = (P, Q, R)$

if $\nabla f = \underline{F}$ then

$$\frac{\partial P}{\partial y} = \partial_2 \partial_1 f \underset{\text{by Clairaut's Thm}}{=} \partial_1 \overbrace{\partial_2 f}^{Q} = \frac{\partial Q}{\partial x}$$

The other two equalities can be proved in a similar way.

8. (Stewart 16.3/36) Use the previous exercise to show that the line integral

$$\int_C y\,\mathrm{d}x + x\,\mathrm{d}y + xyz\,\mathrm{d}z$$

is not dependent of path.

$$\underline{F}(x,y,z) = (y, x, xyz) = (P(x,y,z), Q(x,y,z), R(x,y,z))$$

For example: $\frac{\partial Q}{\partial z} = 0 \neq xz = \frac{\partial R}{\partial y} \Rightarrow \underline{F}$ is not conservative

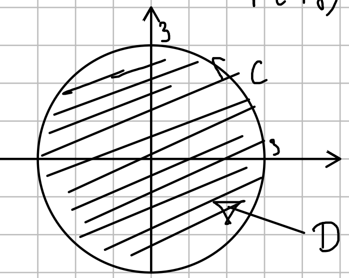
9. (Stewart 16.4/ Example 2, Exercise 9,17) Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

a) $\oint_C \left(3y - e^{\sin(x)}\right)\,\mathrm{d}x + \left(7x + \sqrt{y^4+1}\right)\,\mathrm{d}y$ where C is the positively oriented circle defined by the equation $x^2+y^2 \leq 9$.

b) $\int_C \left(y + e^{\sqrt{x}}\right)\,\mathrm{d}x + \left(2x + \cos(y^2)\right)\,\mathrm{d}y$ where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

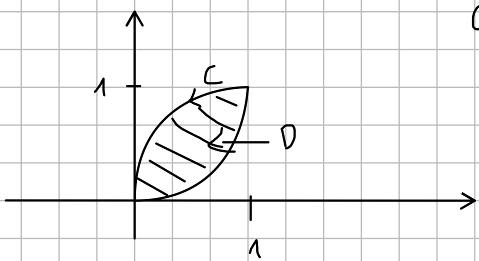
c) $\int_C (y - \cos(y))\,\mathrm{d}x + x\sin(y)\,\mathrm{d}y$ where C is the circle defined by the equation $(x-3)^2 + (y+4)^2 \leq 4$ oriented clockwise.

a) $$\int_C \Big(\underbrace{3y - e^{\sin(x)}}_{P(x,y)}\Big)dx + \Big(\underbrace{7x + \sqrt{y^4+1}}_{Q(x,y)}\Big)dy \underset{\text{Green's formula}}{=} \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, d(x,y) = \int_D 7-3 \, d(x,y) = 4 \cdot \underbrace{A(D)}_{9\pi} = 36\pi$$



$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$$

b) $$\int_C y + e^{\sqrt{x}}\, dx + 2x + \cos(y^2)\, dy \underset{\text{Green's formula}}{=} \int_D 2-1 \, d(x,y) = \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy\, dx = \int_0^1 x^{\frac{1}{2}} - x^2 \, dx = \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3}\right]_{x=0}^{1} = \frac{1}{3}$$



c) $$\int_C \overbrace{y - \cos(y)}^{P(x,y)}\, dx + \overbrace{x\sin(y)}^{Q(x,y)}\, dy \underset{\text{Green's formula}}{\overset{!}{=}} -\int_D \sin(y) - 1 - \sin(y)\, d(x,y) = A(D) = 4\pi$$